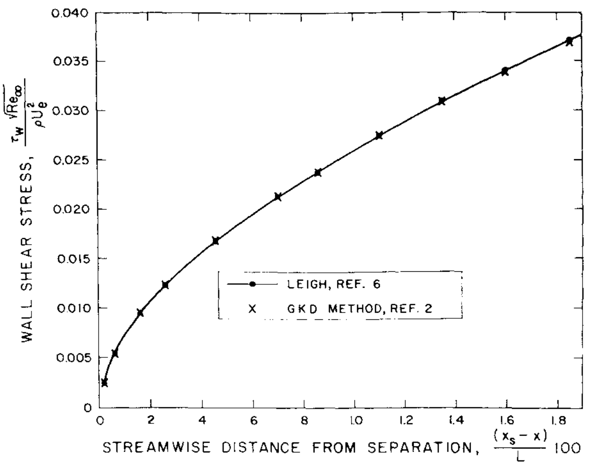

Fig. 2 Wall shear stress distribution in the neighborhood of separation for the linearly retarded flow.

The function α (x) is an additional unknown that must become zero as x approaches the separation point for (5) to have the proper behavior there. The requirement that $1/\Phi \sim u^{-1/2}$ at the separation point was known to Dorodnitsyn.[4] A form of the approximating function similar to (5), but without the logarithmic term,

$$(1/\Phi)_N \sim 1/\{[\alpha + u]^{1/2}(1 - u)\} \qquad (6)$$

was employed to solve first-order problems by Neilson et al.[5] and Bethel.[2] In Ref. 2 it was found that the form (6) could only be employed in the retarded flow region, and a form similar to (2) but without the logarithmic term

$$(1/\Phi)_N \sim \{(1 - u)\}^{-1} \qquad (7)$$

was utilized in the accelerated flow region. The latter form of the approximating function adequately handled mildly retarded flows but did not satisfactorily handle strongly retarded flows and is incapable of predicting separation.[2]

A result obtained in the vicinity of the separation point is given below to illustrate the applicability of the modified form of the approximating function to regions of adverse pressure gradient. This result is for the first-order boundary-ayer equations and was obtained with the form of the approximating function (6). The wall shear stress distribution in the neighborhood of the separation point for the linearly retarded flow, $U = 1 - \frac{1}{8} x/L$, is compared with that obtained by Leigh[6] in Fig. 2. The approximate results were obtained with four coefficients by the so-called GKD (Galerkin-Kantorovich-Dorodnitsyn) method of Ref. 2 and are accurate to within less than 2%.

In conclusion, this comment has attempted to show why Devan was unable to obtain satisfactory results in regions of adverse pressure gradient. A modified formulation for adverse pressure gradient regions is derived and results obtained with an allied method are given to illustrate the success of the reformulation in regions of adverse pressure gradient.

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Reply by Author to H. E. Bethel

Leroy Devan*
University of Kansas, Lawrence, Kansas

I WOULD like to thank H. E. Bethel for his Technical Comment[1] on my paper.[2] The modifications given by Bethel evidently will improve first-order approximate boundary-layer computations in regions of retarded flows.

The numerical methods developed in Ref. 2 were intended to be applied to slightly retarded flows. Table 1 of Ref. 2 is extended further into the retarded flow regime in Table 1 of the present reply. It is seen that for an odd number of undetermined coefficients the approximate method is high, and for an even number it is low. An increase in the number of strips improves the numerical values of the shear stress parameter F. It is anticipated that increasing the number of strips will be uneconomical in the use of computing time for separated flows and indeed will be incapable of predicting separation. The author agrees with the basic conclusions of Bethel; a reformulation of the shear stress variation is required for retarded flow regions.

The logarithmic form of the shear stress in Ref. 2 and in Ref. 1 [Eq. (2)] is required for second-order computations. Accelerated flow computations indicate that the logarithmic form is superior to the simple 1-u form given in Eq. (7) of Ref. 1. It is anticipated that improved first-order profiles for adverse pressure regions also will improve second-order computations.

Table 1 First-order solution for Rankine source body in adverse pressure region

y	s	M_1	$F(0, y)^a$	$F(0, y)^b$	$F(0, y)^c$	$F(0, y)^d$	$F(0, y)^e$
2.16	3.511	−0.051	0.4070	0.4076	0.4077	0.4070	0.4070
2.22	3.759	−0.073	0.3680	0.3692	0.3688	0.3679	0.3679
2.28	4.037	−0.092	0.3294	0.3317	0.3301	0.3292	0.3292
2.34	4.352	−0.107	0.2914	0.2954	0.2922	0.2911	0.2911
2.40	4.712	−0.119	0.2545	0.2610	0.2551	0.2541	0.2541
2.46	5.128	−0.127	0.2193	0.2294	0.2193	0.2188	0.2187
2.52	5.618	−0.132	0.1864	0.2018	0.1853	0.1859	0.1854
2.58	6.203	−0.133	0.1570	...	0.1534	0.1564	0.1550
2.64	6.920	−0.130	0.1327	...	0.1242	0.1320	0.1287
2.70	7.819	−0.123	0.1157	...	0.0983	0.1157	0.1081
2.74	8.560	−0.117	0.1099	...	0.0832	0.1119	0.0986
2.78	9.460	−0.110	0.1093	...	0.0726	0.1163	0.0940
2.82	10.575	−0.100	0.1140	...	0.0589	0.1300	0.0953
2.86	11.997	−0.092	0.1242	...	0.0522	0.1494	0.1044
2.88	12.868	−0.085	0.1315	...	0.0498	0.1595	0.1112
2.90	13.880	=0.079	0.1402	...	0.0485	0.1690	0.1193

[a] Computations by A. M. O. Smith and D. Clutter of Douglas Aircraft Co., Long Beach, Calif.
[b] 3 strips.
[c] 4 strips.
[d] 5 strips.
[e] 6 strips.

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* Assistant Professor of Engineering, Department of Mechanics and Aerospace Engineering. Member AIAA.

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Effect of Wave Reflections on the Unsteady Hypersonic Flow over a Wedge

K. J. Orlik-Rückemann*
National Aeronautical Establishment, Ottawa, Canada

The case of a thin wedge oscillating in pitch in inviscid hypersonic flow was considered independently in two relatively recent papers.[1,2] In both papers, pressure distributions and stability derivatives were obtained from an unsteady analysis based on the hypersonic small-disturbance theory. In Ref. 1 the results are given in terms of closed expressions valid for low reduced frequencies, whereas in Ref. 2 the presentation (which at the same time applies to a more general type of motion) is valid for arbitrary frequencies, but has the form of infinite series. As compared to earlier developed approximate methods, such as the simple-wave and the strong-shock piston theories, the two solutions represent a more general approach, which takes into account not only the presence of bow shock wave but also the influence of secondary unsteady waves reflecting from the bow shock. Unfortunately, although both solutions reduce correctly to the previous approximate results, their prediction of the effect of reflected waves differs radically, even having an opposite sign. It is the purpose of this note to resolve these differences by pointing out and correcting certain analytical and algebraical errors in Ref. 1, and by reducing, for the case of low reduced frequencies, the infinite series of Ref. 2 to closed-form expressions. It is shown also that the results of both papers can be reduced to identical algebraic expressions of a very simple form. Symbols of Refs. 1 and 2 are used, wherever applicable, throughout this note.

In Ref. 1, the unsteady one-dimensional equations of continuity, momentum, and entropy are linearized in terms of the unsteady perturbation quantities $\bar{p}$ and $\bar{v}$, and the solutions are presented in the form

$$\bar{p}/\rho_2 a_2 v_2 = \tfrac{1}{2}[f(\xi) + g(\eta)] \quad (1)$$

$$\bar{v}/v_2 = \tfrac{1}{2}[f(\xi) - g(\eta)] \quad (2)$$

where f and g are linear functions of the characteristic parameters $\xi = \tau(v_2 + a_2) - y$ and $\eta = \tau(v_2 - a_2) - y$. Boundary conditions at the shock ($y = \tau V_0$) are obtained by perturbing the shock wave relations and are given by

$$\tfrac{1}{2}[f(\alpha\tau) + g(\beta\tau)] = 2\mu M\bar{V}/V_0 \quad (3)$$

$$\tfrac{1}{2}[f(\alpha\tau) - g(\beta\tau)] = \lambda\bar{V}/V_0 \quad (4)$$

where $\alpha = \xi/\tau = v_2 + a_2 - V_0$ and $\beta = \eta/\tau = v_2 - a_2 - V_0$. The analytical error in Ref. 1 arises in the application of these boundary conditions; this, unfortunately, affects the entire analysis. The corrected version is summarized below:

From Eqs. (3) and (4) we obtain at the shock

$$f(\alpha\tau) = (2\mu M + \lambda)\bar{V}(\tau)/V_0 = 2G\bar{V}(\tau)/V_0 \quad (5)$$

$$g(\beta\tau) = (2\mu M - \lambda)\bar{V}(\tau)/V_0 = 2H\bar{V}(\tau)/V_0 \quad (6)$$

which means that, in general,

$$f(\xi) = 2G\bar{V}(\xi/\alpha)/V_0 \quad (7)$$

$$g(\eta) = 2H\bar{V}(\eta/\beta)/V_0 \quad (8)$$

and

$$\bar{p}/\rho_2 a_2 v_2 = G\bar{V}(\xi/\alpha)/V_0 + H\bar{V}(\eta/\beta)/V_0 \quad (9)$$

$$\bar{v}/v_2 = G\bar{V}(\xi/\alpha)/V_0 - H\bar{V}(\eta/\beta)/V_0 \quad (10)$$

which replaces the corresponding equations in Ref. 1, where $\bar{V}$ was expressed incorrectly as a function of either ξ or η. We now can write the tangency condition at the wedge surface ($y = v_2\tau$) in the form

$$(\theta_0 U_2/v_2)[i\omega(\tau - hc/U_2) + 1]\exp[i(\omega\tau + \phi)] = G\bar{V}(a_2\tau/\alpha)/V_0 - H\bar{V}(-a_2\tau/\beta)/V_0 \quad (11)$$

Following the method of Ref. 1, we assume a power series for $\bar{V}$ and determine it by equating the coefficients of same powers of τ in Eq. (11); we then obtain, from Eq. (9), the unsteady pressure perturbation $\bar{p}_p$ at the wedge surface

$$\frac{\bar{p}_p}{\rho_2 a_2} = \theta_0 U_2 \exp(i\phi) \sum_{r=0}^{\infty} (i\omega)^r \left(\frac{\tau^r}{r!}\right) \times \left(r + 1 - \frac{i\omega hc}{U_2}\right)\frac{(1 + W\delta^r)}{(1 - W\delta^r)} \quad (12)$$

where $\delta = -\alpha/\beta$ and $W = H/G$. Expanding the function of $W\delta^r$ in a geometrical series and introducing exponential functions as the sums of the various infinite series involved, we can write

$$\frac{\bar{p}_p}{\rho_2 a_2} = \theta_0 U_2 \exp(i\omega t)\left[\left(1 - \frac{i\omega hc}{U_2}\right) \times \left\{1 + 2\sum_{r=1}^{\infty} W^r \exp[i\omega\tau(\delta^r - 1)]\right\} + i\omega\tau\left\{1 + 2\sum_{r=1}^{\infty} W^r\delta^r \exp[i\omega\tau(\delta^r - 1)]\right\}\right] \quad (13)$$

Neglecting terms of order $(\omega\tau)^2$ and higher, and introducing $\tau = x/U_2$, we obtain

$$\frac{\bar{p}_p}{\rho_2 a_2 U_2} = \frac{1 + W}{1 - W}\theta + \left\{\left[\frac{1 + 3W\delta}{1 - W\delta} - \frac{2W}{1 - W}\right]\frac{x}{c} - \frac{1 + W}{1 - W}h\right\}\frac{\dot{\theta}c}{U_2} \quad (14)$$

from which we can determine the pitching-moment derivatives,

$$-m_\theta = 2(\rho_2 a_2 U_2/\rho_\infty U_\infty^2)(\tfrac{1}{2} - h)(1 + W)/(1 - W) \quad (15)$$

$$-m_{\dot{\theta}} = 2\frac{\rho_2 a_2}{\rho_\infty U_\infty}\left\{\frac{1}{3}\left[2\frac{1 + W\delta}{1 - W\delta} - \frac{1 + W}{1 - W}\right] - h\frac{1 + W\delta}{1 - W\delta} + h^2\frac{1 + W}{1 - W}\right\} \quad (16)$$

The present analysis differs from that of Ref. 1 by the presence of quantity δ, which in Ref. 1 would be equal to -1. Since $(-m_\theta)$ is independent of δ, it should be identical to the corresponding result in Ref. 1. However, it easily can be seen that $(1 + W)/(1 - W) = L(1 + N)$, whereas in Ref. 1, $(-m_\theta)$ is proportional to $L(1 - N)$. This discrepancy can be traced to an algebraical error in Eq. (18) in Ref. 1. For the same reason, setting $\delta = -1$ in the foregoing equation for $(-m_{\dot{\theta}})$ will not make it identical to the corresponding equation in Ref. 1.

It is interesting to compare the previous results with those based on the method of Ref. 2. In Ref. 3, which is an extension of Ref. 2, each of the lift-force and pitching-moment derivatives is given as the sum of a term representing the approximate piston theory approach and an infinite series, representing the effect of wave reflections. Thus, for in-

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* Head, Unsteady Aerodynamics Laboratory. Associate Fellow AIAA.